A duality-type method for the design of beams

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Abstract

We discuss the nonconvex optimal shape design problem of minimizing the weight of a loaded beam subject to deflection constraints. We associate to it a convex minimization problem which will play the role of a dual. The algorithm we propose has a global character and iterates between the two optimization problems via a so called “resizing rule”.

1 Introduction

We consider the classical example of a beam, with various boundary conditions (see Haslinger and Neittaanmäki [3], Casas [1]):

$$\begin{aligned} (bu^3y'')'' &= f \quad \text{in }]0,1[, & (1)\\ y(0) = y(1) &= 0, & (2)\\ y''(0) = y''(1) &= 0; & (3) \end{aligned}$$

$$\begin{aligned} y(0) &= y'(0) = 0, & (1.2')\\ y''(1) &= (bu^3y'')'(1) = 0; & (1.3') \end{aligned}$$

$$\begin{aligned} y(0) &= y(1) = 0, & (1.2'')\\ y'(0) &= y'(1) = 0; & (1.3'') \end{aligned}$$

where $u(x)$ denotes the thickness of the beam, y is the deflection, $f \leq 0$ is the load, and $b > 0$ is a material coefficient.

The boundary value problem (1) - (3) models a simply supported beam, (1) + (2’), (3’) corresponds to a cantilevered beam, and (1) + (2”), (3”) represents a clamped beam.

A typical optimal shape design problem is to minimize the weight of the beam,

$$\text{Min} \int_0^1 u(x)\, dx \qquad (1.4)$$

subject to various constraints on u, y. Here, we require that the thickness $u \in L^\infty(0,1)$ should stay between some prescribed limits,

$$0 \le a \le u(x) \le M, \quad \text{a.e. in }]0,1[, \tag{1.5}$$

and that the deflection should exceed a given value ($r > 0$):

$$y(x) \ge -r \quad \text{in } [0,1]. \tag{1.6}$$

Other variants will be considered as well. The above formulation is obtained after scaling the variable domain represented by the beam itself onto a fixed domain. This is the *mapping method* due to Murat and Simon [10], and the function describing the initially variable geometry enters in the coefficients of the differential equation (i.e., u is the thickness of the beam). We refer to Haslinger, Neittaanmäki and Tiba [4] or Tiba [13] for other examples of this kind. Let us also mention that there are many alternative approaches to optimal shape design problems; we only quote here the controllability method of Tiba [14], Tiba and Neittaanmäki [16], the classical boundary variation technique Pironneau [12], Haslinger and Neittaanmäki [3], etc. From the point of view of numerical approximation, various finite element schemes have been proposed, and a recent improvement of the convergence properties is due to Hlavàček [6], [7].

Since the mapping $u \mapsto y$, as defined by (1.1)–(1.3), is highly nonlinear, the optimization problem (1.1)–(1.6) is nonconvex. This is one of its main difficulties. In this setting, duality theory is not yet completely clarified, although there are several notable cases and approaches as those of Toland [17], Lindberg [8], Vinter [18] and Young [20], as well as some very good surveys due to Hiriart-Urruty [5], Outrata and Jarusek [11], Männikkö [9].

The method proposed here is a duality-type approach in the sense that we associate with the optimization problem (1.1)–(1.6) another minimization (optimal control) problem which is easier (convex) and will give relevant information for (1.4). However, our idea is different from the previously mentioned work and is not inspired by the convex duality theory. Our starting point is the simple mechanical intuition that if a given thickness $\bar{u}$ is not optimal for (1.1)–(1.6), then the load f may be increased. Therefore, we define a dual-type problem (associated to (1.1)–(1.6) and to $\bar{u}$) where another parameter – the load f – can be varied:

$$\text{Min} \quad \int_0^1 f(x)\,dx, \tag{1.7}$$

$$
\begin{aligned}
(b\bar{u}^3 y'')'' &= f \quad \text{in }]0,1[, && (1.8)\\
y(0) = y(1) &= 0, && (1.9)\\
y''(0) = y''(1) &= 0, && (1.10)\\
f &\leq 0 \quad \text{a.e. in } [0,1], && (1.11)\\
y(x) &\geq -r \quad \text{in } [0,1]. && (1.12)
\end{aligned}
$$

It is understood that the boundary conditions (2'), (3'), or (2"), (3"), can be used alternatively in place of (2), (3).

In the following sections the relationship between the two problems will be explored and an algorithm will be given. A numerical example is thoroughly investigated in Section 4.

Let us also point out that neither the existence of optimal pairs for the problem (1.1)–(1.6) nor for the problem (1.7)–(1.12) can be guaranteed, in general, and will be assumed in the sequel. If in (1.5) some boundedness on u' is required (or in (1.7) some coercivity with respect to f) then existence results are known to hold.

Our method is partially comparable with the Fully Stressed Design (FSD) approach used by engineers which is discussed, for instance, in [2, Ch. 9].

2 The Simply Supported Beam

As it is standard in the literature, we shall denote the problem (1.1)–(1.6) by (P) (primal) and the problem (1.7)–(1.12) by (D) (dual).

Definition 2.1 (a) An admissible control u for (P) is called *extremal* iff for any $\tilde{u} \leq u$ a.e. $[0,1]$, $\tilde{u}$ admissible for (P), it holds $\tilde{u} = u$ a.e. in $[0,1]$.
(b) An admissible control f for (D) is called *extremal* iff for any $\tilde{f} \leq f$ a.e. in $[0,1]$, $\tilde{f}$ admissible for (D), it holds $\tilde{f} = f$ a.e. in $[0,1]$.

Proposition 2.2 If $\bar{u}$ is a local minimizer for (P) and $f \not\equiv 0$, then $\bar{u}$ is extremal for (P).

Proof. Assume that $\bar{u}$ is not extremal for (P), that is, there is some $\tilde{u} \leq \bar{u}$, $\tilde{u} \neq \bar{u}$, which is feasible for (P). Let $\tilde{y}$, $\bar{y}$ be the states associated with $\tilde{u}$, $\bar{u}$, respectively, via (1.1)–(1.3). Then

$$b\tilde{u}^3(\tilde{y} - \bar{y})'' = b\bar{y}''(\bar{u}^3 - \tilde{u}^3). \tag{2.1}$$

Since $f \le 0$, $f \not\equiv 0$, then (1.1), (1.3) show that $b\tilde{u}^3\tilde{y}'' > 0$, $b\bar{u}^3\bar{y}'' > 0$ in $]0,1[$ and (2.1) yields similarly that $\tilde{y} < \bar{y}$ in $]0,1[$.

Let $u_\lambda = \lambda\tilde{u} + (1-\lambda)\bar{u}$, $\lambda \in [0,1]$, and let y_λ denote the corresponding solution to (1.1)–(1.3).

Then

$$\begin{aligned} \tilde{u} \le u_\lambda &\le \bar{u}, \quad \forall \lambda \in [0,1], \text{ a.e. } x \in [0,1], \\ \tilde{u} \ne u_\lambda &\ne \bar{u}, \quad \forall \lambda \in]0,1[, \\ \tilde{y} < y_\lambda &< \bar{y}, \quad \forall \lambda \in]0,1[, \ \forall x \in]0,1[, \end{aligned}$$

that is, u_λ is feasible for (P), for all $\lambda \in [0,1]$.

Notice that, in general, a convex combination of two admissible controls for (P) may be no longer admissible. Here, the admissibility remains valid since $\tilde{u}$, $\bar{u}$ are comparable.

Obviously, $u_\lambda \to \bar{u}$ strongly in $L^\infty(0,1)$ for $\lambda \to 0$, and

$$\int_0^1 u_\lambda(x)dx < \int_0^1 \bar{u}(x)dx, \quad \forall \lambda > 0.$$

This contradicts the local minimum property of $\bar{u}$, and the proof is finished.

Since the problem (D) is convex, it may have only global minimum points, and the statement corresponding to Proposition 2.2 is obvious.

Theorem 2.3 (a) If $f \not\equiv 0$ is extremal for (D) with $a \le \bar{u} \le M$, then $\bar{u}$ is extremal for (P).
(b) If $\bar{u}$ is extremal for (P) and $a = 0$, then f is extremal for (D) if $f \le 0$, $f \ne 0$.

Proof (a) By hypothesis, we see that $\bar{u}$ satisfies the control constraints and (1.12) shows that $\bar{u}$ is admissible for (P).

We notice that if f is extremal for (D) then it follows that there is some $x_0 \in]0,1[$ such that $\bar{y}(x_0) = -r$. Otherwise, there is some $\varepsilon \in]0,r[$ with $\bar{y}(x) \ge -r+\varepsilon$, $\forall x \in [0,1]$, where $\bar{y}$ is the solution to (1.8)–(1.10) corresponding to f (and to $\bar{u}$, in fact). Then

$$\left(b\bar{u}^3\bar{y}'' \cdot \frac{r}{r-\varepsilon}\right)'' = f\frac{r}{r-\varepsilon} \quad \text{in }]0,1[,$$

and $\frac{r}{r-\varepsilon}f$ is admissible for (D). Since $f \not\equiv 0$, we have $\frac{r}{r-\varepsilon}f \le f \le 0$ with $\frac{r}{r-\varepsilon}f \ne f$, which contradicts the extremality of f.

Now, let us prove that $\bar{u}$ is extremal for (P). Assume that $\tilde{u}$ with $\tilde{u} \leq \bar{u}$, $\tilde{u} \neq \bar{u}$, is admissible for (P), and let by $\tilde{y}$, $\bar{y}$ denote the corresponding solutions to(1.1)–(1.3). Then

$$b\tilde{u}^3\tilde{y}'' = b\bar{u}^3\bar{y}'' > 0 \quad \text{in }]0,1[,$$

and therefore $\tilde{y}'' > 0$, $\bar{y}'' > 0$, $\tilde{u} > 0$, $\bar{u} > 0$ in $]0,1[$. That is (see (2.1)),

$$b\tilde{u}^3(\tilde{y} - \bar{y})'' = b\bar{y}''(\bar{u}^3 - \tilde{u}^3) \geq 0, \; \neq 0$$

and we obtain $\tilde{y} < \bar{y}$ in $]0,1[$ which contradicts the admissibility of $\tilde{u}$ and $\bar{y}(x_0) = -r$.

(b) Again, the hypothesis yields that f is admissible for (D). Moreover, if $\bar{u}$ is extremal for (P) and $\bar{y}$ is the associated state, then there is some $x_0 \in]0,1[$ such that $\bar{y}(x_0) = -r$. Indeed, otherwise there would be some $\varepsilon \in]0,r[$ such that $\bar{y}(x) \geq -r + \varepsilon$ for any $x \in [0,1]$. Then

$$\left(b\frac{r-\varepsilon}{r}\bar{u}^3\frac{r}{r-\varepsilon}\bar{y}''\right)'' = f \quad \text{in } \;]0,1[, \tag{2.2}$$

and the pair $\left[\dfrac{r}{r-\varepsilon}\bar{y}, \left(\dfrac{r-\varepsilon}{r}\right)^{\frac{1}{3}}\bar{u}\right]$ is admissible for (P), since $a = 0$. But $\left(\dfrac{r-\varepsilon}{r}\right)^{\frac{1}{3}}\bar{u} \leq \bar{u}$ and $\left(\dfrac{r-\varepsilon}{r}\right)^{\frac{1}{3}}\bar{u} \neq \bar{u}$, since we may assume that $\bar{u} \not\equiv 0$ by $f \not\equiv 0$. This contradicts the extremality of $\bar{u}$, and thus the existence of some $x_0 \in]0,1[$ with the desired property is proved.

Now, if $\hat{f}$ with $\hat{f} \leq f$, $\hat{f} \neq f$, is admissible for (D) and $\hat{y}$ is the associated state, we have

$$(b\bar{u}^3(\bar{y} - \hat{y})'')'' = f - \hat{f} \geq 0,$$

that is, $b\bar{u}^3(\bar{y} - \hat{y})'' < 0$ in $]0,1[$, and $\hat{y}(x) < \bar{y}(x)$, $x \in]0,1[$ which contradicts the admissibility of $\hat{f}$ and $\bar{y}(x_0) = -r$. This shows that f is extremal for (D) and the proof is finished. $\square$

Remarks: 1) The previous proof shows that the extremality of u or f in (P), (D) yields that the associated state is active with respect to the state constraint.

2) We also notice that admissible controls u cannot attain the value 0 in $]0,1[$, even in case b), if $f \not\equiv 0$. This is again a direct consequence of the maximum principle as shown in the above proof.

3) Another simple consequence of the above argument is that in order to obtain an extremal f starting from an admissible one for the problem (D) it is enough to multiply the state equation by an appropriate positive constant, $k = -\dfrac{r}{\min_{x\in[0,1]} y(x)}$, where y is the corresponding state. If $a = 0$, a similar argument may be used for (P), as suggested in (2.2).

Next, we indicate an approximation procedure for the problem (P) in the case that $a = 0$, $M = +\infty$ and $f \not\equiv 0$. An integrability assumption on u^{-3} has to be imposed which will be explained in the next section.

Algorithm 2.4

1. Let $n = 0$, and let u_0 be an admissible control for (P).
2. Solve Min (D_n), corresponding to the coefficient u_n, and denote the minimizer by f_n.
3. If $f_n = f$ (or $f_n - f$ is "small"), then STOP! Otherwise,
4. ("Resizing Step") define $u_{n+1}^3 := u_n^3 \dfrac{g}{g_n}$; and $n := n + 1$, GO TO **2**.

Here, g, g_n are associated to f, f_n, respectively, through $g'' = f$, $g_n'' = f_n$ in $]0,1[$, with zero boundary conditions. Obviously, we have $f_n \not\equiv 0$ and this gives $g_n > 0$ in $]0,1[$.

Moreover, Hopf's lemma implies that $g'(0) > 0, g_n'(0) > 0$, as well as $g'(1) < 0, g_n(1) < 0$. Hence, by l'Hospital's rule, the limits $\lim_{t\to 0+} \dfrac{g(t)}{g_n(t)}$ and $\lim_{t\to 1-} \dfrac{g(t)}{g_n(t)}$ exists and are positive.
Consequently, the functions g/g_n are continuous and everywhere positive on $[0,1]$.

1) The previous results show that each u_n is extremal for (P) since the state constraint will be active. The algorithm has a global character and convergence can be expected only in special cases as the examples show.

2) If $a > 0$ or $M < +\infty$, then Step 4 should include a projection of the control on its constraint set. In Step 1 a local minimization routine for (P) may be added, as well.

3) We notice that in each step of the algorithm a new "dual" problem (D_n) is used, depending on the coefficient u_n.

4) The first three steps of the algorithm give a test for other solution methods for the problem (P) in order to see how accurate the obtained solution is.

5) The stopping criterion is equivalent to the test $u_n = u_{n+1}$ and has the meaning that the algorithm cannot be continued. However, this is not the case with respect to the equality $f_n = f_{n+1}$ which just says that $u_n^2 = u_{n+1} \cdot u_{n-1}$. In the applications other types of stopping criterions are used.

In the final part of this section, we comment briefly on the cantilevered beam, i.e. on the boundary conditions (1.2'), (1.3'). Since a variant of the maximum principle is valid under these Cauchy type conditions (Weinberger and Protter [19, Ch.I]), this case behaves similarly as the simply supported beam. We note that under negative load the state constraint always may become active just in $x = 1$. In the Step 2 of the Algorithm 2.4, it is conceptually possible that $f_n \equiv 0$ in $[s, 1]$ for some $s \in]0, 1[$, although the control constraint has a maximum in 0. If this is the case, then g_n defined by Step 4, will satisfy $g_n \equiv 0$ in $[s, 1]$, and the *resizing* Step 4 will become impossible.

To avoid this situation, one can refine the definition of the dual problem (D_n) by imposing the control constraint

$$f \leq h_n \text{ in } [0, 1]$$

where $h_n < 0$ a.e. in $[0, 1]]$. This can be interpreted as taking the load of the beam itself into account. For instance, we may take $h_n = -\vartheta u_n$, where $\vartheta > 0$ denotes the density of the material.

3 The Clamped Beam

We begin with a comparison result for the solution to the boundary value problem

$$(bu^3 y'')'' = f \quad \text{in }]0, 1[, \tag{3.1}$$
$$y(0) = y(1) = 0, \tag{3.2}$$
$$y'(0) = y'(1) = 0, \tag{3.3}$$

where $f \in L^2(0, 1)$, and where $u \geq 0$ is such that $\frac{1}{u^3} \in L^2(0, 1); b$ is a positive constant. Then the solution y belongs to $W_0^{2,2}(0, 1)$ with $u^3 y'' \in W^{2,2}(0, 1)$, and y is uniquely determined.

Theorem 3.1 If $f \leq 0$ in $[0, 1]$ then $y \leq 0$ in $[0, 1]$.

Proof. We denote $g := bu^3 y''$, which is a concave C^1-mapping in $[0, 1]$. Hence, g may change sign at most twice in $[0, 1]$, that is, it may have at most two distinct roots in $]0, 1[$, unless it is identically zero in some subinterval.

Assume first that g has exactly two roots ξ_1, ξ_2 such that $0 < \xi_1 < \xi_2 < 1$. Then, again by the concavity, g (and y'') is positive in $]\xi_1, \xi_2[$ and negative in $]0, \xi_1[\cup]\xi_2, 1[$. Furthermore, this shows that y is concave in $[0, \xi_1], [\xi_2, 1]$ and convex in $[\xi_1, \xi_2]$ respectively. Taking the boundary conditions (3.2), (3.3) into account, we obtain that $y \leq 0$ in $[0, 1]$. If $g \equiv 0$ in $[\xi_1, \xi_2]$, then $y'' \equiv 0$ in $[\xi_1, \xi_2]$ and thus $y(x) = mx + n$ with suitable $m, n \in \mathbb{R}$, in $[\xi_1, \xi_2]$. We infer that $y'(\xi_1) = y'(\xi_2) = m$, but this contradicts Hopf's lemma (Weinberger and Protter [19, Ch. I], Theorem 2), applied to $[0, \xi_1]$ and $[\xi_1, 1]$ respectively, which gives strictly opposite signs for $y'(\xi_1)$ and $y'(\xi_2)$. Hence this situation is impossible.

Next, assume that g has just one root ξ such that $0 < \xi < 1$. Then, the concavity of g yields that g (and y'') changes sign in ξ or, otherwise, that $g \leq 0$ in $[0, 1]$ and ξ is a maximum point for g. Let us assume first that g (and y'') is positive in $[0, \xi[$ and negative in $]\xi, 1]$. We infer that y is convex in $[0, \xi]$ and concave in $[\xi, 1]$ and the boundary conditions show $y \leq 0$ in $[0, \xi]$, $y \leq 0$ in $[\xi, 1]$. Since y is continuous, it follows $y(\xi) = 0$, and the convexity properties of y imply that $y \equiv 0$ in $[0, 1]$, that is, $g \equiv 0$ in $[0, 1]$, which contradicts the starting assumption. The situation is similar when the opposite signs are assumed.
It remains to consider the case when g has constant sign, (which also includes the second subcase from above). Then, unless y and g vanish identically, we get a contradiction to Hopf's lemma and the boundary condition (3.3). This concludes the proof of the assertion. $\square$

Remark: The only possible situation is that g changes sign twice in $[0, 1]$, unless $g \equiv 0$. This result is not a consequence of sucessive applications of the maximum principle for second order equations, since owing to the boundary conditions, the maximum principle does not apply directly to y''. We now define the problems (P) and (D) as in Section 2 where the boundary conditions are replaced by (1.2"), (1.3"). The duality-type algorithm has the same structure as in 2.4., with the resizing step modified according to the properties of the solution y and of g.

Algorithm 3.2

1. $n = 0$, u_0 admissible for (P);
2. Min (D_n) gives y_n, f_n;
3. If $f_n = f$ (or $f_n - f$ is small), then STOP! otherwise,
4. ("Resizing Step")

α) compute the roots ξ_1^n, ξ_2^n of $g_n = bu_n^3\ y_n''$,

β) define $\tilde{g}_n$ as follows:

i) $\begin{cases} \tilde{g}_n'' = f & \text{in }]0, \xi_1^n[, \\ \tilde{g}_n(\xi_1^n) = 0 \ , \ \tilde{g}_n'(\xi_1^n -) = \tilde{g}_n'(\xi_1^n +); \end{cases}$

ii) $\begin{cases} \tilde{g}_n'' = f & \text{in }]\xi_1^n, \xi_2^n[, \\ \tilde{g}_n(\xi_1^n) = \tilde{g}_n(\xi_2^n) = 0; \end{cases}$

iii) $\begin{cases} \tilde{g}_n'' = f & \text{in }]\xi_2^n, 1[, \\ \tilde{g}_n(\xi_2^n) = 0 \ , \ \tilde{g}_n'(\xi_2^n +) = \tilde{g}_n'(\xi_2^n -) \end{cases}$

γ) resize u_n by $u_{n+1}^3 = u_n^3 \dfrac{\tilde{g}_n}{g_n}$;

and $n := n+1$, GO TO 2.

Remarks: 1) One has to solve part ii) first, and afterwards parts i) and iii), in Step 4, β). Since $f_n \le 0$, it follows that g_n has the properties derived in the proof of Theorem 3.1, and it is very easy to see that $\tilde{g}$ is C^1 and has the same sign as g_n. Therefore $u_{n+1} \ge 0$, and the significance of the resizing Step 4 is that (as before)

$$(bu_{n+1}^3 y_n'')'' = f$$

(we return to the right-hand side f and iterate).
2) The resizing step gives in fact the solution of an identification problem, since we know for the boundary value problem (1.1), (1.2"), (1.3") that y_n should be the solution, and we have to find the coefficient u_{n+1}. From this point of view the above algorithm may be directly extended to very general situations.
3) If ξ_1^n, ξ_2^n are known then g_n may be alternatively defined by $g_n'' = f_n$ in $]0,1[$ and $g_n(\xi_1^n) = g_n(\xi_2^n) = 0$. If $f \neq 0$ a.e. in $]0,1[$, then Hopf's lemma implies that in a neighborhood of ξ_1^n, ξ_2^n it holds

$$\begin{aligned} \tilde{g}_n(x) &\sim \tilde{k}_i(x - \xi_i^n) \quad , \quad i = 1,2, \\ g_n(x) &\sim k_i(x - \xi_i^n) \quad , \quad i = 1,2, \end{aligned}$$

with $\tilde{k}_i \neq 0$. Thus, $\dfrac{g_n}{\tilde{g}_n} \in L^\infty(0,1)$ for any n, and hence $\dfrac{1}{u_{n+1}^3} \in L^2(0,1)$ if $\dfrac{1}{u_n^3} \in L^2(0,1)$. Therefore, the algorithm will preserve the integrability properties of the control.
4) It is not clear whether a comparison result of the type "$0 < u_1(x) \le u_2(x)$ a.e. $[0,1] \Rightarrow y_1(x) \le y_2(x)$" is valid. Therefore, we state the definition of extremality implicity on the state, and not directly on the control as in the previous section.

Definition 3.3 An admissible control u for (P) is called *extremal* iff the state constraint is active for the corresponding solution y to (3.1), (3.2), (3.3).

Proposition 3.4 Any local minimizer of (P) is extremal for (P).

Proof. Otherwise, there is some $\lambda > 1$ such that $\lambda y(x) \geq -r$ in $[0,1]$ and the pair $[\lambda^{-\frac{1}{3}} u, \lambda y]$ is admissible, since

$$(b[\lambda^{-\frac{1}{3}} u]^3 \lambda y'')'' = f.$$

But $\lambda^{-\frac{1}{3}} u < u$ in $]0,1[$ and, for $\lambda \searrow 1, \lambda^{-\frac{1}{3}} u$ belongs to any neighborhood of u. This contradicts the fact that u is a local minimum for (P). □

Corollary 3.5 For $n \geq 1$, the sequence $\{u_n\}$ generated by the Algorithm 3.2 consists of extremals for the problem (P).

Proof. We just have to confirm that the optimal pair $[y_n, f_n]$ for (D_n) is active with respect to the state constraint. Indeed, otherwise there is some $\lambda > 1$ such that the pair $[\lambda y_n, \lambda f_n]$ is admissible for (D_n) and, in addition $\int_0^1 \lambda f_n(x) dx < \int_0^1 f_n(x) dx$, since $f_n \leq 0, f_n \neq 0$. This contradicts the optimality of f_n. □

4 Numerical Experiments

We have only studied the case of a simply supported beam. To this end we divided the interval $[0,1]$ into 32 equal elements and we considered two basic examples, namely $f(x) = -48$, $x \in [0,1]$, and $f(x) = -48$ in $[0, \frac{1}{2}]$, $f(x) = 0$ in $]\frac{1}{2}, 1]$, which we shall denote subsequently as (P_1) and (P_2). The initial guess was always $u_0(x) = 3$, $x \in [0,1]$, which is admissible for both (P_1) and (P_2), as can easily be verified by hand if we put $b = 1$ and assume the constraints:

$$0 \leq u(x) < \infty, \tag{4.2}$$

$$\text{(a)} \;\; y(x) \geq -0.33, \qquad \text{(b)} \;\; y(x) \geq -0.5. \tag{4.3}$$

While the constraint (4.2) never became active by the very nature of the problem (P), the constraint (4.3) (a) or (b) was active at any time, since the algorithm was generating extremals in each iteration except the initial one. For the treatment of (4.3), we used a standard penalization technique, i.e. we added to the cost functional in (P) or in (D) the term

$$\frac{1}{\varepsilon}\int_0^1 (y+r)_-^2\,(x)dx, \tag{4.4}$$

where $\varepsilon > 0$ is "small" and $r = -0.33$ or $r = -0.5$. The typical choice was $\varepsilon = 10^{-5}$ or $\varepsilon = 10^{-8}$, and this approach allowed for some minor violations of the state constraint in several points of the grid. The experiments have shown that the result depends on the choice of ε, which is an undesirable perturbation of the original algorithm, and more refined techniques for dealing with state constraints like the augmented Lagrangian method or the variational inequality approach (see Tiba and Neittaanmäki [15]) will have to be tested as well.

We have all the time compared our algorithm with the standard descent method provided by the NAG library applied directly to (P) (which we also use in Step 2 for problem (D)). As the following table shows, the volume of the beam provided by our algorithm was at any time slightly smaller than the one obtained by the standard descent method.

Table 4.1. Volume of the beam ($\varepsilon = 10^{-5}$)

Problem	r	Standard method	Dual method
P_1	-0.33	1.0759	1.0313
P_1	-0.5	0.9368	0.9277
P_2	-0.33	0.8301	0.8256

Slightly larger violations of the state constraint were observed in the dual approach. The stopping test which was used is related to the descent property of the algorithm. Since the problems are nonconvex and the proposed algorithm has a global character, some oscillations of the cost may occur, and we stop when the descent property is violated for the first time.

We have also tested the behaviour of the algorithm after the descent property has been violated. In the case of the problem (P_1) we obtained convergence in all experiments, that is, the criterion $f_n = f$ (or, equivalently, $u_n = u_{n-1}$) was fulfilled after a finite number of steps. However, the obtained limit u_n is just an extremal for (P), not optimal, and is hard to interpret. Moreover, this property is no longer valid for the nonsymmetric case (P_2). In general, the thickness of the beam was decreased in three to four steps until the optimum is achieved; then another mechanical property is put into evidence: due to the boundary conditions (simply supported beam), the force acting on the boundary elements may be very large, and the resizing Step 4 gives a very thin beam near the boundary points. In order to satisfy the deflection constraints, on the contrary, the beam has to be thicker in the interior region, and so the total volume of the beam is increased in this second part of the algorithm. The resulting beam has the property that its coincidence set (where the state constraint is "approximately" active) becomes bigger and bigger, almost the whole interval $[0,1]$ in some experiments.

Suggested by this observation, we have introduced a perturbation $u_{n+1} \to u_{n+1} + \delta$ in Step 4. We used $\delta = 0.3$, and this trick avoided that the algorithm degenerated in its second phase. So, only normal small oscillations of the cost around the minimal value were observed, and repeating the example (P_2) in this way, we obtained a decrease of the volume of the beam down to 0.7956. The stopping test here was simply a preassigned number of iterations, namely 20. The volume 0.8035 was obtained already in the fifth iteration.

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